LII. Fig. 28.

$ABC = BEF$. $HRQ = ACJ$. $ARH = HKA$ is equivalent to $AKIJ + FDQ$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

LIII. Fig. 28.

$AMNH$ is equivalent to $ACLH$ is equivalent to $ACIK$.

So, $MBFN$ is equivalent to $BEDC$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

Wipper.

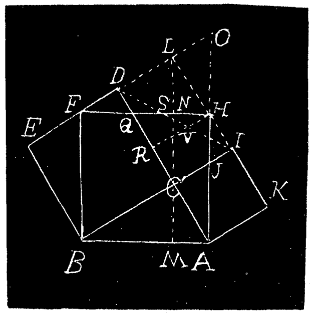


Fig. 28.

LIV. Fig. 28.

$CLOJ$ is equivalent to $CLHA$ is equivalent to $ACIK$.

$BFLC$ is equivalent to $BEDC$.

But $ABFH$ is equivalent to $BFOJ$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

Hoffmann, 1800.

LV. Fig. 28.

$ABFH + BEF + FLH + HKA$ is equivalent to $ACIK + BEDC + ABC + CIL + CLD$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

LVI. Fig. 28.

$ABC = BEF$. $ICD = AKH$ is equivalent to $AKIJ + FDQ$.

$SVH = SQD$, and $VHT = IJT$.

$\therefore$ By properly combining and substituting, $ABFH$ is equivalent to $ACIK + BEDC$.

LVII. Fig. 28.

$RDLH = ACIK$. $ARH = BEF$. $ABC = HFL$.

$\therefore$ $ABFH$ is equivalent to $ACIK + BEDC$.

[To be Continued.]

EUCLIDEAN GEOMETRY WITHOUT DISPUTED AXIOMS.

By G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire.

(*a*)

PROPOSITION I. *If two straight lines in the same plane be perpendicular to the same straight line they are parallel.*

Prove by Axiom 11, and I, 27.*

*These and the subsequent numbers refer to the Book and Proposition in Todhunter's Euclid.

(b)

Proposition II. *From or through a given point in a straight line only one perpendicular to that line can be drawn in the same plane.*

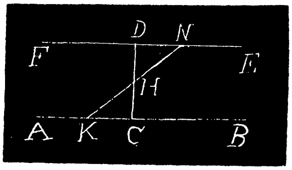
Proof. If there could be two, there would be two unequal right angles, which is impossible by Axiom 11.

(c)

Proposition III. *If two parallel straight lines be joined by a common perpendicular, any straight line which bisects the perpendicular and meets the two parallels is itself bisected by the perpendicular.*

Let AB be a straight line. Take any point in it as C and erect the perpendicular CD (I, XI). At D erect the perpendicular DE (I, 11) and extend it to F (Postulate 2). Then FE is parallel to AB (a).

Now bisect DC in H, (I, 10), take any point in AC as K and join KH, (Postulate I). On DE cut off DN equal to KC, (I, 2), and join HN, (Postulate 1). Therefore the two triangles KCH and DHN are equal to each other (I, IV). Therefore KH equals HN. Again, since the two triangles KCH and DHN are equal, the angle DHN equals the angle KHC, being homologous angles. The angles KHC and KDH are together equal to two right angles (I, 13). Therefore since the angle DHN equals the angle KHC, the angles DHN and KHD are together equal to two right angles, and therefore KH and HN form one and the same straight line (I, 14). Therefore, since K is any point in AB, any straight line which bisects the perpendicular joining two parallel straight lines is bisected by the perpendicular.

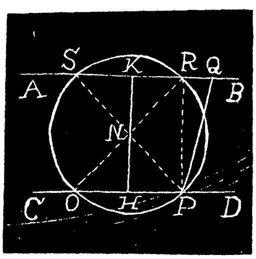


Corollary. If two parallel straight lines be joined by a common perpendicular, any straight line meeting the parallels and bisecting the perpendicular cuts off equal distances on the parallels on opposite sides of the perpendicular.

(d)

Proposition IV. *If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also.*

Proof. Let CD be a straight line. Then from any point in it as H draw HK perpendicular to CD, and in the same manner draw AB perpendicular to KH (I, 11). Then AB and CD are parallel (a). Take any point in one of the parallels as P in CD and suppose PQ be drawn perpendicular to CD. Then will PQ be perpendicular to AB also. For cut off $HO=HP$ (I, 2), bisect HK at N (I, 10), and draw PS and OR through N. Then $NO=NP$ (I, 4). But $SN=NP$ and $NO=NR$ (c). Therefore $NS=NO=NP=NR$ (Axiom 1). Therefore, similarly, $OH=HP=KR=SK$ (c, Corollary). With N as a center and NO as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points O, P, R, and S. Draw PR. The angle NHO is greater than the an-



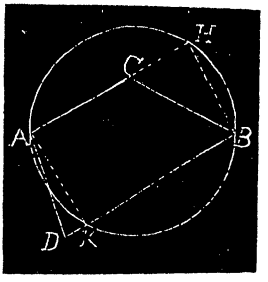
gle NPH (I, 16), therefore the angle NHP is greater than the angle NPH, and therefore NP is greater than NH (I, 19). Therefore the circumference of this circle will intersect the two parallel lines in the points O, P, R, and S. The angle OPR is a right angle (III, 31), and therefore RP is perpendicular to CD. But QP is by hypothesis perpendicular to CD, therefore PQ and PR cannot form two separate lines (b). Therefore PQ, if properly drawn must be identical with PR. But the angle SRP is a right angle (III, 31) and therefore PQ is perpendicular to AB.

Q. E. D.

(e)

Proposition V. *If the vertex of an angle subtended by the diameter of a circle is between the center and circumference, the angle is greater than a right angle; and if the vertex is without the circle the angle is less than a right angle.*

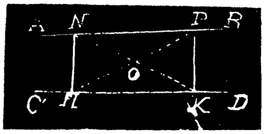
Proof. Let AKH be a circle, AB a diameter of that circle, and let it subtend the two angles ACB and D, the vertex of the former being within, and that of the latter without, the circle. Extend AC to the circumference at point H, and join HB and KA (Postulate 1). Therefore the angles H and AKB are right angles (III, 31). Therefore the angle ACB is greater than angle H and angle D is less than angle AKB (I, 16).



(f)

Proposition VI. *If two parallel straight lines be joined by two common perpendiculars, these two perpendiculars are equal to each other.*

Proof. Let AB and CD be two parallel straight lines and let NH and PK be perpendicular to CD, then are they also perpendicular to AB (d). Join NK and HP (Postulate 1). Bisect HP (I, 10), then with the middle point of HP as a center and one-half HP as a radius describe a circle (Postulate 3). The circumference of this circle will obviously pass through the points H and P. It must also pass through N and K, otherwise the angles HNP and HKP would not be right angles (e). Again, bisect NK (I, 10) and with its middle point as a center and one-half NK as a radius describe another circle (Postulate 3). The circumference of this circle will also pass through the points N, K, P, and H for the same reason as the one above. Therefore these circumferences will coincide with one another (III, 10). Therefore there can be but one center point which being in both the lines NK and HP must be at the point of intersection O. Therefore the two triangles NOH and POK are equal to each other (I, 4), and therefore NH equals PK.



Q. E. D.

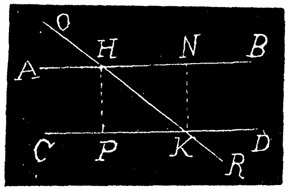
Corollary. The intercepts on two parallel straight lines by two common perpendiculars are equal to each other.

For, the triangles NOP and HOK are equal to each other (I, 4). Therefore NP is equal to HK, being homologous sides of two equal triangles.

(g)

PROPOSITION VII. *If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, etc.* (I, 29).

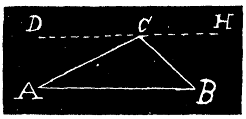
PROOF. Let the straight line OR fall on the two parallel straight lines AB and CD, meeting them in points H and K respectively. Then the angles BHK and CKH shall be equal to one another.



From H draw HP perpendicular to CD, and from K draw KN perpendicular to AB (I, 12). Then HP is also perpendicular to AB and KN is also perpendicular to CD (d). Therefore HP equals NK (f), and HN equals PK (f Corollary). Therefore the two triangles HPK and HNK are equal to each other (I, 8), and therefore the angle NHK equals the angle HKP, being homologous angles of two equal triangles. Q. E. D.

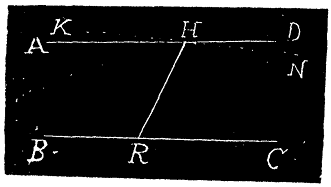
PROPOSITION VIII. *The sum of the angles of every plane triangle is equal to two right angles.*

PROOF. Let ABC be any plane triangle, then the sum of the angles A, B, and C is equal to two right angles. Through one of its vertices as C draw DH parallel to AB (I, 31). Then the angles A and DCA are equal to one another (g), as are also the angles B and HCB for the same reason. But the sum of the angles DCA, ACB, and BCH is equal to two right angles (I, 13). Therefore the sum of the angles, A, B, and ACB must equal two right angles. Q. E. D.

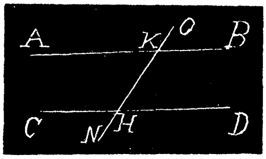


PROPOSITION IX. *Through a given point without a given straight line only one line can be drawn parallel to the given line.*

PROOF. Let BC be a straight line and H a point without. Draw AD through H parallel to BC (I, 31). Then no other line can be drawn through H parallel to BC. If possible suppose KN drawn through H parallel to BC. Then since the angles KHR and AHR are each equal to the angle HRC (g), they are equal to each other (Axiom 1), a part to the whole which is impossible. Therefore KN cannot be parallel to BC. Q. E. D.



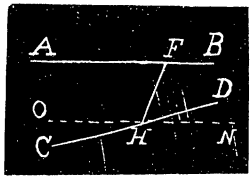
PROPOSITION X. *If a straight line fall on two parallel straight lines, the sum of the two interior angles on the same side of that line shall be equal to two right angles.*



PROOF. Let the straight line ON fall on the two parallel straight lines AB and CD. Then the sum of the two angles AKH and CHK is equal to two right angles. For, the sum of the two angles CHK and KHD is equal to two right angles (I, 13) and the angle AKH equals the angle KHD (g). Therefore, substituting the latter for the former we have the sum of the two angles AKH and CHK equal to two right angles. Q. E. D.

PROPOSITION XI. *If a straight line meet two straight lines so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.* Euclid, Axiom 12.

PROOF. Let the straight line FH meet the two straight lines AB and CD, making the two angles BFH and FHD together less than two right angles, then AB and CD shall meet, if continually produced, on that side of FH towards B and D. Since the angles BFH and AFH are together equal to two right angles, they must be greater than the sum of the two angles BFH and FHD. Therefore, the angle AFH must be greater than the angle FHD. Hence, draw the line ON through H making the angle FHN equal to the angle AFH (I, 23). Then ON is parallel to AB (I, 27). Therefore CD cannot be parallel to AB (i), and therefore CD and AB must meet if sufficiently produced. Since the sum of the angles AFH and FHO equals two right angles (j), the sum of the angles AFH and FHC must be greater than two right angles. Therefore AB and CD cannot meet on that side of FH toward A and C for then we should have a triangle the sum of whose angles would be greater than two right angles which is impossible by (h). Therefore they must meet on that side of FH toward B and D.



Q. E. D.

ZERO, INFINITESIMALS, INFINITY, AND THE FUNDAMENTAL SYMBOL OF INDETERMINATION.

By GEORGE LILLEY, Ph. D., Professor of Mathematics, State University, Washington.

The following is an outline of the method I use in explaining to the student in algebra how zero is used as a multiplier and a divisor, and how infinitesimals and infinity are used as divisors; also, interpretations of the results obtained by their use.

If we multiply a by a number that decreases by 1 each time beginning with any number, as $+4$, and continue the multiplication until -4 is reached, each product will decrease by a. Thus,

a	a	a	a	a	a	a	a
$+4$	$+3$	$+1$	*zero*	-1	-2	-3	-4
$+4a$	$+3a$	$+a$,*	zero,	$-a$	$-2a$	$-3a$	$-4a$,

where zero is a constant number and obtained by subtracting *any* number from itself, as, $a-a=⓪$, ⓪ representing *absolute* zero.

Evidently a multiplied by zero is one a less than a multiplied by $+1$, or